



Grade 7/8 Math Circles

October 11/12/13/17, 2022

Mathematical Logic - Solution

1. Let P stand for the statement “I will eat spaghetti for dinner” and Q for the statement “I will eat spaghetti for lunch.” What English sentences are represented by the following expressions?
 - a. $\neg(P \wedge \neg Q)$.
 - b. $\neg P \wedge \neg Q$.
 - c. $\neg P \vee \neg Q$.

Solution:

- a. I will not both eat spaghetti for dinner and not eat it for lunch.
- b. I will not eat spaghetti for dinner and I will not eat spaghetti for lunch.
- c. I will not eat spaghetti for dinner or I will not eat spaghetti for lunch. (Note that \vee means **inclusive** or.)

2. Identify the premises and conclusions of the following deductive arguments and analyze their logical forms. Do you think the reasoning is valid? (Note: “either” means **exclusive** or.)
 - a. Min and Claire won’t both win the math prize, Claire will win either the math prize or the science prize. Min will win the math prize. Therefore, Claire will win the science prize.
 - b. Either Alex or Claire is telling the truth. Either Min or Claire is lying. Therefore, either Alex is telling the truth or Claire is lying.

Solution:

- a. Let’s assume that all of our premises are true. Then Min must win the math prize. Since Min and Claire won’t both win the math prize, Claire will not win the math prize. Since Claire must win either the math prize or the science prize, she must win the science prize - this is our conclusion. This conclusion was *forced* on us when we assume that all of our premises are true. Hence, this reasoning is valid.
- b. If Alex and Min are lying (so, not telling the truth) and Claire is telling the truth, we have that all of our premises are true. However, if Alex is lying and Claire is telling the truth, our conclusion is false. Recall our definition for a valid deductive



reasoning (when we have all premises true, the conclusion must be true as well).
This reasoning is invalid.

3. Find simpler logical formulas equivalent to:

a. $\neg P \vee \neg(Q \wedge P)$.

b. $\neg((P \wedge \neg Q) \vee \neg(\neg P \vee Q))$.

Solution:

a. $\neg P \vee \neg(Q \wedge P) \equiv \neg P \vee (\neg Q \vee \neg P)$ (De Morgan's law)

$$\equiv \neg P \vee (\neg P \vee \neg Q) \quad (\text{Commutative law})$$

$$\equiv (\neg P \vee \neg P) \vee \neg Q \quad (\text{Associative law})$$

$$\equiv \neg P \vee \neg Q \quad (\text{Idempotent law})$$

$$\equiv \neg(P \wedge Q) \quad (\text{De Morgan's law})$$

b. $\neg((P \wedge \neg Q) \vee \neg(\neg P \vee Q)) \equiv \neg((P \wedge \neg Q) \vee (\neg\neg P \wedge \neg Q))$ (De Morgan's law)

$$\equiv \neg((P \wedge \neg Q) \vee (P \wedge \neg Q)) \quad (\text{Double Negation law})$$

$$\equiv \neg(P \wedge \neg Q) \quad (\text{Idempotent law})$$

4. Make a truth table for $P \vee (Q \vee \neg P)$ and $P \wedge \neg(Q \vee \neg Q)$.

Solution:

- $P \vee (Q \vee \neg P)$.

| P | Q | $\neg P$ | $Q \vee \neg P$ | $P \vee (Q \vee \neg P)$ |
|-----|-----|----------|-----------------|--------------------------|
| T | T | F | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | F | T | T | T |

Logical formulas that are always *true*, such as $P \vee (Q \vee \neg P)$, are called **tautologies**.

- $P \wedge \neg(Q \vee \neg Q)$.



| P | Q | $\neg Q$ | $Q \vee \neg Q$ | $\neg(Q \vee \neg Q)$ | $P \wedge \neg(Q \vee \neg Q)$ |
|-----|-----|----------|-----------------|-----------------------|--------------------------------|
| T | T | F | T | F | F |
| T | F | T | T | F | F |
| F | T | F | T | F | F |
| F | F | T | T | F | F |

Logical formulas that are always *false*, such as $P \wedge \neg(Q \vee \neg Q)$, are called **contradictions**.

5. Find simpler logical formulas equivalent to:

(Hint: What are the truth values of $P \vee \neg P$, $P \wedge \neg P$, (a statement that is always true) $\wedge P$, (a statement that is always false) $\vee P$?)

- $P \vee (Q \wedge \neg P)$.
- $\neg(\neg(P \wedge (Q \vee \neg Q)) \vee Q)$
- $\neg(P \vee (Q \wedge \neg R)) \wedge Q$.

Solution: Please try Question 4. before solving this question. $P \vee \neg P$ is a tautology and $P \wedge \neg P$ is a contradiction. We also have (tautology) $\wedge P \equiv P$, (tautology) $\vee P \equiv$ (tautology), (contradiction) $\vee P \equiv P$, and (contradiction) $\wedge P \equiv$ (contradiction).

$$\begin{aligned} \text{a. } P \vee (Q \wedge \neg P) &\equiv (P \vee Q) \wedge (P \vee \neg P) && \text{(Distributive law)} \\ &\equiv (P \vee Q) \wedge (\text{tautology}) \\ &\equiv P \vee Q \end{aligned}$$

$$\begin{aligned} \text{b. } \neg(\neg(P \wedge (Q \vee \neg Q)) \vee Q) &\equiv (\neg\neg(P \wedge (Q \vee \neg Q)) \wedge \neg Q) && \text{(De Morgan's law)} \\ &\equiv ((P \wedge (\text{tautology})) \wedge \neg Q) && \text{(Double Negation law)} \\ &\equiv P \wedge \neg Q \end{aligned}$$



$$\begin{aligned} \text{c. } \neg(P \vee (Q \wedge \neg R)) \wedge Q &\equiv (\neg P \wedge \neg(Q \wedge \neg R)) \wedge Q && \text{(De Morgan's law)} \\ &\equiv (\neg P \wedge (\neg Q \vee \neg\neg R)) \wedge Q && \text{(De Morgan's law)} \\ &\equiv (\neg P \wedge (\neg Q \vee R)) \wedge Q && \text{(Double Negation law)} \\ &\equiv \neg P \wedge ((\neg Q \vee R) \wedge Q) && \text{(Associative law)} \\ &\equiv \neg P \wedge ((\neg Q \wedge Q) \vee (R \wedge Q)) && \text{(Distributive law)} \\ &\equiv \neg P \wedge ((\text{contradiction}) \vee (R \wedge Q)) \\ &\equiv \neg P \wedge (R \wedge Q) \end{aligned}$$

6. Five friends with different ages are side by side wearing different coloured shirts, drinking different juice, and talking about different deals they got during the Black Friday sales. They each got different discount on the deals. Find out who bought the laptop.
- A. The man drinking the Orange juice is exactly to the right of the man who got the 70% discount.
 - B. Keith is 45 years old.
 - C. The man who bought the TV is exactly to the left of the man wearing the Red shirt.
 - D. At the third position is the man who got the 50% discount.
 - E. Keith is next to the man wearing the White shirt.
 - F. The 25-year-old man is somewhere between the 35-year-old man and the 40-year-old man, in that order.
 - G. The man drinking Apple juice bought the Smartphone.
 - H. The 30-year-old man is exactly to the left of the man that bought the Beard trimmer.
 - I. Sean is the youngest (25-year-old).
 - J. The man that got the 40% discount is exactly to the right of the man who bought the Beard trimmer.
 - K. Keith is next to the 35-year-old man.
 - L. Eugene is 40 years old.
 - M. Sean is wearing the Black shirt.
 - N. At the fourth position is the man who got the biggest discount (80%).
 - O. Dustin got 60% off.
 - P. The man drinking the Lemon juice is exactly to the right of the man drinking the Grape juice.



- Q. Keith bought a Game console.
- R. The man who got the 80% discount is exactly to the left of the man who is wearing the Blue shirt.
- S. The man drinking Grape juice bought the Beard trimmer.
- T. The man wearing the Black shirt is somewhere to the right of Keith.
- U. The man that bought the Smartphone is next to the man wearing the Black shirt.

Taken from brainzilla (<https://www.brainzilla.com/>)

Solution:

| | Man #1 | Man #2 | Man #3 | Man #4 | Man #5 |
|----------|--------|--------|--------|--------|--------|
| Name | | | | | |
| Shirt | | | | | |
| Deal | | | | | |
| Discount | | | | | |
| Age | | | | | |
| Juice | | | | | |

- Using simple clues D and N, we know that Man #3 got the 50% discount and Man #4 got the 80% discount.
- By R, Man #5 is wearing the Blue shirt.
- By H and J, we know that Man #4 bought the Beard trimmer and this gives that Man #3 is the 30-year-old man and that Man #5 got the 40% discount.
- By S and P, Man #4 and #5 drink Grape juice and Lemon juice, respectively.
- By F and K, we know that Man #4 and Man #5 are 25 years old and 40 years old, respectively.
- By L and I, Man #5 is Eugene and Man #4 is Sean.
- By M, Man #4 is wearing the Black shirt.
- By G and U, we know that Man #3 bought the Smart phone and drinks Apple juice.
- By A, Man #1 got the 70% discount and Man #2 drinks the Orange juice.
- By O, Man #2 is Dustin who got the 60% discount.
- By B and Q, Man #1 is Keith, the 45-year-old, who got the Game console.



- By E, Dustin is wearing the White shirt.
- By C, we know that Man #2 bought the TV and Man #3 is wearing the Red shirt.
- Laptop is the only remaining option for Man #5.

| | Man #1 | Man #2 | Man #3 | Man #4 | Man #5 |
|----------|--------------|--------|-------------|---------------|--------|
| Name | Keith | Dustin | | Sean | Eugene |
| Shirt | | White | Red | Black | Blue |
| Deal | Game console | TV | Smart phone | Beard trimmer | |
| Discount | 70% | 60% | 50% | 80% | 40% |
| Age | 45 | | 30 | 25 | 40 |
| Juice | | Orange | Apple | Grape | Lemon |

Therefore, we conclude that Eugene (Man #5) bought the laptop.



7. Sudoku - Easy

| | | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1 | | | 2 | | | 3 | | 6 |
| 9 | | | | | 7 | 4 | 2 | |
| | 7 | | | | | | | 1 |
| | | | | | 4 | 5 | | 7 |
| | 1 | | | 9 | | 6 | | |
| 2 | 4 | 5 | | | 8 | | | |
| | 5 | 8 | 7 | 3 | 2 | | | |
| | 6 | | | | | | 3 | |
| | | | 9 | | | | | 4 |

Taken from PUZZLES.ca (<https://www.puzzles.ca/sudoku/>)

Solution:

| | | | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 1 | 8 | 4 | 2 | 5 | 9 | 3 | 7 | 6 |
| 9 | 3 | 6 | 8 | 1 | 7 | 4 | 2 | 5 |
| 5 | 7 | 2 | 4 | 6 | 3 | 8 | 9 | 1 |
| 6 | 9 | 3 | 1 | 2 | 4 | 5 | 8 | 7 |
| 8 | 1 | 7 | 3 | 9 | 5 | 6 | 4 | 2 |
| 2 | 4 | 5 | 6 | 7 | 8 | 9 | 1 | 3 |
| 4 | 5 | 8 | 7 | 3 | 2 | 1 | 6 | 9 |
| 7 | 6 | 9 | 5 | 4 | 1 | 2 | 3 | 8 |
| 3 | 2 | 1 | 9 | 8 | 6 | 7 | 5 | 4 |



8. Sudoku - Medium

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | | | | | | | | |
| | | | 6 | | | 1 | 2 | 5 |
| | 4 | | | | | 6 | | |
| 7 | | | | 4 | 9 | 8 | | 1 |
| 3 | 6 | | 7 | | | | 4 | 9 |
| | | | | | 8 | | 5 | |
| | 2 | 1 | 8 | | 5 | 3 | | |
| 5 | | 9 | | | | 4 | | |
| 6 | 3 | | | | 2 | | | |

Taken from PUZZLES.ca (<https://www.puzzles.ca/sudoku/>)

Solution:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | 1 | 6 | 5 | 2 | 7 | 9 | 3 | 4 |
| 9 | 7 | 3 | 6 | 8 | 4 | 1 | 2 | 5 |
| 2 | 4 | 5 | 9 | 1 | 3 | 6 | 8 | 7 |
| 7 | 5 | 2 | 3 | 4 | 9 | 8 | 6 | 1 |
| 3 | 6 | 8 | 7 | 5 | 1 | 2 | 4 | 9 |
| 1 | 9 | 4 | 2 | 6 | 8 | 7 | 5 | 3 |
| 4 | 2 | 1 | 8 | 7 | 5 | 3 | 9 | 6 |
| 5 | 8 | 9 | 1 | 3 | 6 | 4 | 7 | 2 |
| 6 | 3 | 7 | 4 | 9 | 2 | 5 | 1 | 8 |



9. Sudoku - Hard

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | | | | | | | | 9 |
| | | | 4 | | 3 | | | |
| 7 | | 4 | 6 | | | | | 5 |
| 9 | | | | 2 | | | 6 | |
| | | 5 | 8 | 3 | | | 4 | |
| | | 8 | | 1 | | 3 | | |
| | 8 | 3 | | | | 6 | | |
| 6 | | 9 | | | 5 | 2 | | 4 |
| | | | | | 9 | | | |

Taken from PUZZLES.ca (<https://www.puzzles.ca/sudoku/>)

Solution:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 6 | 2 | 7 | 8 | 4 | 1 | 9 |
| 8 | 9 | 1 | 4 | 5 | 3 | 7 | 2 | 6 |
| 7 | 2 | 4 | 6 | 9 | 1 | 8 | 3 | 5 |
| 9 | 3 | 7 | 5 | 2 | 4 | 1 | 6 | 8 |
| 1 | 6 | 5 | 8 | 3 | 7 | 9 | 4 | 2 |
| 2 | 4 | 8 | 9 | 1 | 6 | 3 | 5 | 7 |
| 5 | 8 | 3 | 7 | 4 | 2 | 6 | 9 | 1 |
| 6 | 1 | 9 | 3 | 8 | 5 | 2 | 7 | 4 |
| 4 | 7 | 2 | 1 | 6 | 9 | 5 | 8 | 3 |